Review on Vectors

• Reading Assignments

H. Anton and C. Rorres, *Elementary Linear Algebra* (Applications Version), 8th edition, John Wiley, 2000 (3.1-3.4, hard copy).

J. Pricipe et al., *Neural and Adaptive Systems: Fundamentals Through Simulations*, (Appendix A: Elements of Linear Algebra and Pattern Recognition, hard copy).

K. Kastleman, *Digital Image Processing*, Prentice Hall, (Appendix 3: Mathematical Background, hard copy).

F. Ham and I. Kostanic. *Principles of Neurocomputing for Science and Engineering*, Prentice Hall, (Appendix A: Mathematical Foundation for Neurocomputing, hard copy)

• Other Books

B. Kolman and D. Hill, *Introductory Linear Algebra with Applications*, 2nd edition, Prentice Hall, 2001.

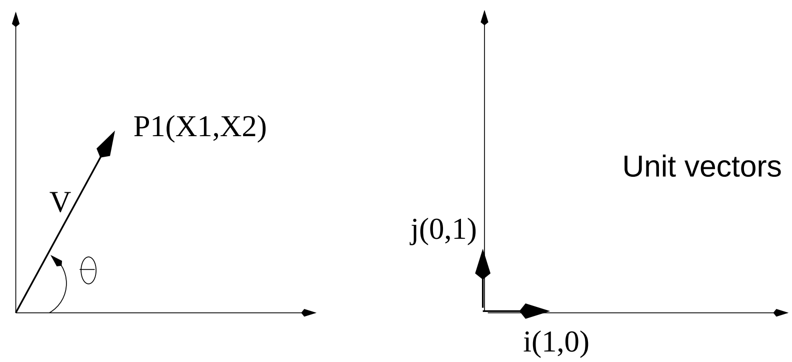
L. Johnson, R. Riess, and J. Arnold, *Introduction to Linear Algebra*, 4th edition, Addison Wesley, 1998.

Review on Vectors

• 2-D Vectors

- Are represenred geometrically as directed line segments.

- Have two attributes: *magnitude* and *direction*



<u>representation:</u> $v = (x_1, x_2)$

<u>magnitude:</u> $||v|| = \sqrt{x_1^2 + x_2^2}$

<u>direction:</u> $\theta = \tan^{-1}(x_2/x_1)$

• Unit vectors

- Any vector with magnitude equal to one.

- Examples are the unit vectors in the x and y directions: $i = (1, 0),\ j = (0, 1)$

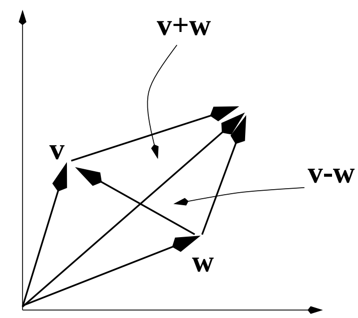
<u>Vector normalization:</u> $\bar{v} = \frac{v}{||v||} = (\frac{x_1}{||v||}, \frac{x_2}{||v||})$

<u>Example:</u>

$$v = (2, 5),\ ||v|| = \sqrt{2^2 + 5^2} = \sqrt{29}$$

$$\bar{v} = (\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}})$$

• Vector Arithmetic



Addition: $v + w = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

Subtraction: $v - w = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$

Scalar multiplication: $av = (ax_1, ax_2)$

• n-dimensional vectors

- An *n*-dimensional vector v and its transpose v^T can be written as:

$$v = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \qquad v^T = [\ x_1 \ x_2 \cdots x_n]$$

- The *Euclidean norm* of a vector is defined as the scalar

$$||v|| = \sqrt{v^T v} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

• Inner (dot) product

- Given two vectors $v = (x_1, x_2, \ldots, x_n)$ and $w = (y_1, y_2, \ldots, y_n)$:

$$v.\,w = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \text{ or}$$

$$v.\,w = [\ x_1 \ x_2 \cdots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = v^T w$$

- Special case ($v = w$):

$$v.v = [x_1 \ x_2 \ \cdots \ x_n] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots + x_n^2 = ||v||^2$$

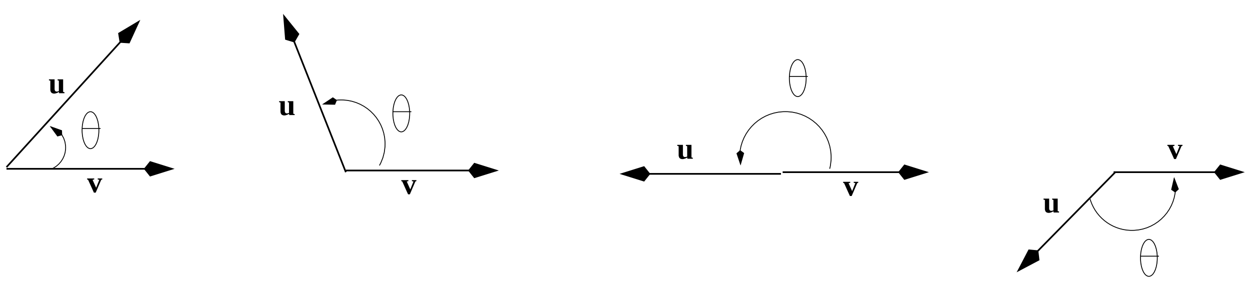
- **The outer product**

$$wv^T = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix} [\ x_1 \ x_2 \ \cdots \ x_n] = \begin{bmatrix} y_1x_1 & y_1x_2 & \ldots & y_1x_n \\ y_2x_1 & y_2x_2 & \ldots & y_2x_n \\ \ldots & \ldots & \ldots & \ldots \\ y_nx_1 & y_nx_2 & \ldots & y_nx_n \end{bmatrix}$$

- *Warning*: the inner product is a **scalar**, while the outer product is a **matrix** !

- **Geometrically-based definition of the dot product**

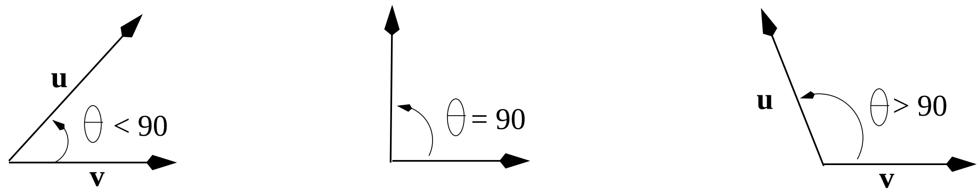
$u.v = ||u|| \ ||v|| \ cos(\theta)$, $0 \leq \theta \leq \pi$ (θ is the smaller angle between u,v)



- **Properties of the dot product**

- $v.(u + aw) = v.u + a(v.w)$

- The sign of $u.v$ depends on the $cos(\theta)$



cos (θ) > 0 | cos (θ) = 0 | cos (θ) < 0

u . v > 0 | u . v = 0 | u . v < 0

• Orthonormal vectors

- A set of vectors $x_1, x_2, \ldots, x_n$ is *orthonormal* if

$$x_i^T x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

• Gram-Schmidt orthogonalization procedure

- Any basis $(x_1, x_2, \ldots, x_n)$ can be converted to an orthonormal basis $(o_1, o_2, \ldots, o_n)$ using the *Gram-Schmidt* orthogonalization procedure.

(1) Make $o_1 = x_1$

(2) Take the second vector x_2 and subtract from it the part that lies along the direction x_1

$$o_2 = x_2 - ao_1, \text{ where } a = \frac{x_2 . o_1}{o_1 . o_1}$$

(3) Continue the process to obtain all the vectors:

$$o_k = x_k - \sum_{i=1}^{k-1} \frac{x_k . o_i}{o_i . o_i} o_i$$